Revised Canonical Quantum Gravity via the Frame Fixing

Simone Mercuri

Giovanni Montani

ICRA—International Center for Relativistic Astrophysics
Dipartimento di Fisica (G9),
Università di Roma, "La Sapienza",
Piazzale Aldo Moro 5, 00185 Rome, Italy.
e-mail: mercuri@icra.it, montani@icra.it

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Abstract

We present a new reformulation of the canonical quantum geometrodynamics, which allows to overcome the fundamental problem of the *frozen formalism* and, therefore, to construct an appropriate Hilbert space associate to the solution of the restated dynamics. More precisely, to remove the ambiguity contained in the Wheeler-DeWitt approach, with respect to the possibility of a $(3 + 1)$-splitting when the space-time is in a quantum regime, we fix the reference frame (i.e. the lapse function and the shift vector) by introducing the so-called kinematical action; as a consequence the new super-Hamiltonian constraint becomes a parabolic one and we arrive to a Schrödinger-like approach for the quantum dynamics.
In the semiclassical limit our theory provides General Relativity in the presence of an additional energy-momentum density contribution coming from no longer zero eigenvalues of the Hamiltonian constraints; the interpretation of these new contributions comes out in natural way as soon as it is recognized that the kinematical action can be recasted in such a way it describes a pressureless, but, in general, non geodesic perfect fluid.

1 Introduction

The first convincing attempt to extend the methods of quantum field theory toward the quantization of the gravitational field was proposed in 1967 by B. DeWitt [7]. DeWitt, in his approach, presented the implementation of the standard canonical method to quantize the Hamiltonian constraints; as well known, such a procedure leads to limiting features which induced to search for more appropriate reformulation of the problem. Until now, we do not have a completely satisfactory theory which unifies the fundamental principles of the quantum mechanics with those of General Relativity, though many different approaches exist: a review of the key difficulties associated with the matching of these two basic theories, especially with respect to the concept of time in quantum gravity, is presented in [14] (for the more recent loop quantum gravity and spin foam approaches, see respectively [31] and [25]).
All these different approaches have lead to develop two main classes of theories: one in which the time variable is determined after the quantization procedure, and another one associated with the choice of the "time" before implementing the quantum dynamics. We stress it exists a correlation between these two different points of view and the consideration that, those quantization procedures, which preserve the covariance of the dynamics, are inconsistent with any notion of space-time slicing; indeed, if the whole metric tensor is in a full quantum regime, then we may separate space-like objects from time-like ones only, at most in the limiting sense of expectation values. In this line of thinking (the incompatibility between a covariant quantization and a (3+1)-splitting), since the canonical method of quantization relies on the notion of an Hamiltonian function and of its conjugate time variables, it can be applied to gravity only when the diffeomorphism invariance is broken.

A fully covariant approach appears well-stated in a path integral formulation which relies on the Lagrangian function; this covariant point of view, initially addressed in [11], [13], has found a promising development in the recent issue of the spin foam formalism [25], in which the notion of space-time continuity is replaced by appropriate discrete microstructures.

The DeWitt's approach presents exactly the problem to require the full covariance of the dynamics and preserve the $(3 + 1)$-splitting of the space-time this procedure allows to obtain the Hamiltonian constraints connected to the gauge invariance of the gravitational theory (i.e. the covariance under the space-time difffeomorphisms); the canonical quantization of the system is then obtained by applying the usual correspondence between the dynamical variables and the quantum operators. When translating the constraints on a quantum level, we get from the super-Hamiltonian the so-called Wheeler-DeWitt equation (WDE) [23], [17], [18], [14] and, from the super-momentum, the invariance of the state function under the 3-diffeomorphisms (J. A. Wheeler gave an important contribution to clarify the role of a class of 3-geometries as the fundamental variable of the theory).

The WDE consists of a functional approach in which the states of the gravi-

tational system are represented by a wave functional taken on the 3-geometries and it reflects the invariance of the quantum dynamics under time displacements; thus, as a consequence, we get loss a real time dependence of the wave functional, i.e. the so-called *frozen formalism*, the most limiting outcoming features of the WDE approach [8].

In fact, due to the frozen formalism, in the WDE no general procedure to turn the space of its solutions into an Hilbert one exists; so any appropriate general notion, either of functional probability either of an "internal" time variable Nevertheless, it is worth noting how, this approach, when applied in the early cosmology problem, [16], [12], in which exists a good internal time variable, the volume of the Universe, seems to be rather expressive, especially with respect to the mechanism underlying the achievement of a classical Universe after the Planckian era [15].

Over the last ten years the canonical quantum gravity has found its best improvement in the reformulation of the constraints problem in terms of the *Ashtekar's variables*, leading to the *loop quantum gravity* theory [5], [30], [31], which overcomes many of the limits of the WDE, in particular it solves the ambiguities about the construction of an Hilbert space, and has now a wide diffusion. However, we remark that the loop quantum gravity contains the same ground limits of the WDE, since the $(3 + 1)$-splitting of the space-time and the full covariance requirements live here till together. But there are reliable expectations for a link between loop quantum gravity and the spin foam approach which, being the latter based on a pure 4-dimensional framework, could provide some predictivity even to the former.

Our point of view is quite different, we think that, as stressed in [24], many of the shortcomings of the WDE approach are unchangeable to a fundamental implicit *ansatz,* on which this theory is based: the possibility to speak of an Arnowitt-Deser-Misner (ADM) formalism [1], [2], [3], [4], [23], when splitting the notion of space from that one of time in a quantum regime. Indeed, independently of the approach we are considering, the notion of a quantum space-time has to be associated to the possibility of knowing metric information only in the sense of expectation values; since to speak of a space-like hypersurface we need to say its normal field is time-like, then the $(3 + 1)$ procedure make sense if translated in terms of quantum projecting operators. But, in the canonical approaches, the splitting is performed before quantizing and is just in such a arbitrariness that our criticism arises. In view of saving the quantum theory covariance, the most rigorous approach seems to be a projecting operator scheme which restores on a quantum level the notions of space and time. Nevertheless the canonical approach may acquire a precise meaning as soon as we give a physical interpretation to the space-time slicing even on a quantum level. To this end we need a physical entity able to distinguish intrinsically the nature of the time; this entity, performing a kind of "measure" on the quantum system, should have a certain degree of classicality. From a more applicative point of view, we should search for a procedure able to ensure the existence of a time-like normal field for any choice of the coordinates system in the quantum space-time. We show that it is possible to joint together these two (concep-

tual and applicative) scenarios, by an appropriate study of what happens in the canonical formalism when we fix the reference frame during the quantization scheme.

Indeed, as shown in [24], fixing the slicing (in other words the reference frame) we are able, ab initio, to distinguish between space-like and time-like geometrical objects, so we can implement the ADM formalism to rewrite the action in the canonical form. But to fix the slicing we have to choose a particular value for the lapse function N and the shift vector N^i, and this is equivalent to loose the Hamiltonian constraints and, with them, the canonical procedure of quantization.
The quantum field theory on a curved background [6], gives us a fundamental indication about how to restore the canonical constraints: we have to reparameterize the action by expressing it through the use of a generic coordinates system; the way toward a correct reparameterization is provided by the so-called *kinematical action* [18], which allows to achieve a satisfactory structure for the quantum constraints. In the case we extend this method to the canonical quantum gravity, it is possible to get a "time" dependence of the wave functional, with respect either to a one parameter family of hypersurfaces, either to a real time variable appearing on a smeared formulation (in this case it assumes a clear physical meaning strictly connected with the physical interpretation of the kinematical term, on a classical as well as a quantum level).

The aim of our paper is to extend the already existing (revised canonical quantum) theory, as presented in [24], in order to eliminate a restriction on the super-momentum term of the kinematical action, (which is taken equal to zero after the variation), with all its classical limit implications.
More precisely, in this paper we renfource the point of view that to restore the canonical constraints after to have fixed the reference frame, it is necessary to introduce the kinematical action, which can be interpreted as the action of a *dust reference fluid*, which interacts with a "gravito-electromagnetic-like" field (GEM). In the classical framework the presence of the dust is the physical consequence of fixing the slicing, in other words it is just the (materialized) reference fluid, in which we describe the dynamics. The idea that, fixing the reference frame, a kind of reference fluid appears, was introduced by the works of Kuchař et al. [19], [20], [21] and [22], (an interesting discussion about the material nature of a reference frame in classical and quantum gravity associated to the problem of physical observable is provided in [28], [29]).

Instead, on the quantum framework, the presence of the dust reflects the no longer vanishing of the eigenvalues of the quantum super-Hamiltonian and super-momentum operators; in particular, these eigenvalues are strictly connected to the energy density of the dust and to the GEM field it fills via the influence exerted on the world lines, (such identifications come when taking the classical limit of the quantum dynamics).

To conclude, it is worth stressing that the presence of such a dust fluid shows its effects only in those systems which have undergone through a quantum phase in their evolution and which later reached a classical limit. Thus we expect that it be possible to observe such effects, due to the presence of the dust, in the

outcomings of the cosmological problem.

Section 2 is dedicated to the Hamiltonian formulation of General Relativity [23], [32], [31] and, in particular, we rewrite the Einstein's equations in terms of the canonical variables, without taking into account the constraints, with the aim of comparing them to the mean values dynamics predicted by our revised quantum theory.

Section 3 is entirely devoted to the physical interpretation of the kinematical action, as viewed on a classical level, which is one of the building blocks of the theory, because it allows to understand the real essence of our approach, anticipating the outcoming of the quantum dynamics.

In Section 4 we develop the quantum theory, starting from the classical action (with the additional kinematical term) and arriving to the functional equations; we show that it is possible to turn the space of the solutions of the time dependent or "Schrödinger-like" equation into an Hilbert one, with a functional norm which allows us to define a notion of probability. We then face the eigenvalues problem and the semiclassical limit, which gets light on the physical meaning of quantizing in a gauge fixing framework.

In section 5 we discuss an Ehrenfest approach to the mean value of the operator corresponding to the classical observables; in the dynamical equations for the mean values there appear additional terms in comparison with the usual classical one, which are due to the particular normal ordering required by the existence of an Hilbert space and they go to zero in the classical limit.

2 Hamiltonian formulation of Einstein's theory

To obtain the Hamiltonian constraints, which are the starting point for the canonical quantization of gravity, we have to write the Einstein-Hilbert action into a (3 + 1) formulation. To this aim, we have to perform a slicing of the 4-dimensional space-time, on which a metric tensor $g_{\mu\nu}$ is defined.

We consider a space-like hypersurface having a parametric equation $y^\rho = y^\rho\left(x^i\right)$ (Greek indices run from 0 to 3, while Latin ones run from 1 to 3) and in each point we define a 4-dimensional vector base composed by its tangent vectors $e^\mu_i = \partial_i y^\mu$ and by the normal unit vector n^μ; as just defined, these vectors base satisfy, by construction, the following relations

$$g_{\mu\nu}e^\mu_i n^\nu = 0, \qquad g_{\mu\nu}n^\mu n^\nu = -1. \tag{1}$$

Now if we deform this hypersurface through the whole space-time, via the parametric equation $y^\rho = y^\rho\left(t, x^i\right)$, we construct a one-parameter family of space-like hypersurfaces slicing the 4-dimensional manifold; thus each component of the adapted base, acquiring a dependence on the time-like parameter t, becomes a vector field on the space-time.

Let us introduce the deformation vector $N^\mu = \partial_t y^\mu\left(t, x^i\right)$, which connects two points, with the same spatial coordinates, on neighboring hypersurfaces (i.e.

corresponding to values of the parameter t and $t+dt$).
This vector field can be decomposed with respect to the base (n^μ, e^μ_i), obtaining the following representation:

$$N^\mu = \partial_t y^\rho = N n^\mu + N^i e^\mu_i. \tag{2}$$

Where N and N^i are, respectively, the lapse function and the shift vector, so this expression is known as lapse-shift decomposition of the deformation vector.

It is easy to realize how the space-like hypersurfaces are characterized by the following 3-dimensional metric tensor $h_{ij} = g_{\mu\nu} e^\mu_i e^\nu_j$. Since the hypersurface is deformed through space-time, it changes with a rate, which taken with respect to the label time t, can be decomposed into its normal and tangential contributions

$$\partial_t h_{ij} = -2N k_{ij} + 2\nabla_{(j} N_{i)}, \tag{3}$$

where $N_i = h_{ij} N^j$, the covariant derivative is constructed with the 3-dimensional metric and $k_{ij} = -\nabla_i n_j$ denotes the extrinsic curvature.

Now we define the co-base vectors $\left(n_\mu, e^i_\mu\right)$, as follows

$$n_\mu = g_{\mu\nu} n^\nu, \qquad e^i_\mu = h^{ij} g_{\mu\nu} e^\nu_j, \tag{4}$$

where h^{ij} is the inverse 3-metric: $h^{ij} h_{jk} = \delta^i_k$. By the second of the above relation we obtain $e^i_\mu e^\mu_j = \delta^i_j$.

The explicit expression for the 4-metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ assume, in the system $\left(t, x^i\right)$, respectively, the form:

$$g_{\mu\nu} = \begin{pmatrix} N_i N^i - N^2 & N_i \\ N_i & h_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{N^2} & \dfrac{N^i}{N^2} \\ \dfrac{N^i}{N^2} & h^{ij} - \dfrac{N^i N^j}{N^2} \end{pmatrix}. \tag{5}$$

Moreover, the normal vector n^μ has the following components $\left(\dfrac{1}{N}, -\dfrac{N^i}{N}\right)$, and this implies that the covariant normal vector be $n_\mu = (-N, 0)$; below we will use to indicate the components of the vectors in the system $\left(t, x^i\right)$, with Greek barred indices as: $\bar{\mu}$, $\bar{\nu}$, $\bar{\rho}$..... We also note that in this system of coordinates, the square root of the determinant of the metric tensor assumes the form $\sqrt{-g} = N\sqrt{h}$.

It is possible to show that the Einstein-Hilbert action can be rewritten as follows [4], [32], [31]:

$$S = \int\limits_{\Sigma^3 \times \Re} dt d^3x N\sqrt{h} \left({}^{(3)}R + k_{ij} k^{ij} - k^2\right), \tag{6}$$

which is the most appropriate to construct the "ADM action" for the gravitational field.
Now, defining the conjugate momenta to the dynamical variables, which are the

component of the 3-metric tensor, we can rewrite the gravitational action in its Hamiltonian form. The gravitational Lagrangian L^g does not contain the time derivative of the lapse function N and of the shift vector N^i, so their conjugate momenta are identically zero and the Lagrangian is said singular. Summarizing, we have for the conjugate momenta:

$$p^{ij}\left(t,x^i\right)=\frac{\partial L^g}{\partial\left(\partial_t h_{ij}\right)}=\sqrt{h}\left(k^{ij}-kh^{ij}\right), \tag{7}$$

$$\pi\left(t,x^i\right)=\frac{\partial L^g}{\partial\left(\partial_t N\right)}=0, \qquad \pi_i\left(t,x^i\right)=\frac{\partial L^g}{\partial\left(\partial_t N^i\right)}=0. \tag{8}$$

By the above definition, we can perform the Legendre dual transformation and, with few algebra, then obtaining the below final form for the gravitational action [31]:

$$S^g\left(h_{ij},p^{kl},N,N^a,\pi,\pi_b\right)=\int\limits_{\Sigma^3\times\Re} dtd^3x\left\{p^{ij}\partial_t h_{ij}+\pi\partial_t N+\pi_k\partial_t N^k\right. \tag{9}$$

$$\left.-\left(\lambda\pi+\lambda^j\pi_j+NH^g+N^iH_i^g\right)\right\} \tag{10}$$

where the so called super-Hamiltonian H^g and super-momentum H_i^g, read respectively as

$$H^g=G_{ijkl}p^{ij}p^{kl}-\sqrt{h}{}^{(3)}R, \qquad H_i^g=-2\nabla_j p_i^j, \tag{11}$$

where (using geometrical units) $G_{ijkl}=\frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl}+h_{il}h_{jk}-h_{ij}h_{kl}\right)$ is the so-called super-metric.

Now, before calculating the other dynamical equations, we want to add to this picture, also a matter field, which, for simplicity, is represented by a self-interacting scalar field ϕ. This lead us to the following expression for the action of the gravitational and matter field:

$$S^{g\phi}=\int\limits_{\Sigma^3\times\Re} dtd^3x\left\{p^{ij}\partial_t h_{ij}+\pi\partial_t N+\pi_k\partial_t N^k+p_\phi\partial_t\phi\right.$$

$$\left.-\left(\lambda\pi+\lambda^j\pi_j+N\left(H^g+H^\phi\right)+N^i\left(H_i^g+H_i^\phi\right)\right)\right\} \tag{12}$$

where the Hamiltonian terms H^ϕ and H_i^ϕ read explicitly as:

$$H^\phi=\frac{1}{2\sqrt{h}}p_\phi^2+\frac{\sqrt{h}}{2}h^{ij}\partial_i\phi\partial_j\phi+\sqrt{h}V\left(\phi\right) \qquad H_i^\phi=p_\phi\partial_i\phi \tag{13}$$

and $V\left(\phi\right)$ denotes the self-interaction potential energy.

Varying the action (12) with respect to the Lagrange multipliers λ and λ_i, we obtain the first class constraints:

$$\pi = 0, \qquad \pi_k = 0; \tag{14}$$

to assure that the dynamics is consistent, i.e. the Poisson parenteses between the constraints and the Hamiltonian be zero, we have to require that the second class constraints

$$H^g + H^\phi = 0, \qquad H^g_i + H^\phi_i = 0, \tag{15}$$

be satisfied.

Moreover varying the action with respect the two conjugate momenta π and π_i, we obtain the two equations

$$\partial_t N = \lambda, \qquad \partial_t N^i = \lambda^i, \tag{16}$$

which ensure that the trajectories of the lapse function and of the shift vector in the phase space are completely arbitrarily.

The action (12) has to be varied with respect to all the dynamical variables and this gives us the Hamiltonian equations for the scalar and the gravitational field, which take the form:

$$\frac{d}{dt}h_{ab} = 2NG_{abkl}p^{kl} + 2\nabla_{(a}N_{b)}, \tag{17}$$

$$\begin{aligned}
\frac{d}{dt}p^{ab} = {} & \frac{1}{2}N\frac{h^{ab}}{\sqrt{h}}\left(p^{ij}p_{ij} - \frac{1}{2}p^2\right) - \frac{2N}{\sqrt{h}}\left(p^{ai}p^b_i - \frac{1}{2}pp^{ab}\right) + \\
& - N\sqrt{h}\left({}^{(3)}R^{ab} - \frac{1}{2}{}^{(3)}Rh^{ab}\right) + \\
& + \sqrt{h}\left(\nabla^a\nabla^b N - h^{ab}\nabla^i\nabla_i N\right) + \\
& - 2\nabla_i\left(p^{i(a}N^{b)}\right) + \nabla_i\left(N^i p^{ab}\right) + \\
& + \frac{N}{4\sqrt{h}}h^{ab}p^2_\phi - \frac{N}{2}\sqrt{h}h^{ab}\left(\frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + V(\phi)\right),
\end{aligned} \tag{18}$$

$$\frac{d}{dt}\phi = \frac{N}{\sqrt{h}}p_\phi + N^i\partial_i\phi, \tag{19}$$

$$\begin{aligned}
\frac{d}{dt}p_\phi = {} & N\sqrt{h}h^{ij}\partial_i\partial_j\phi + \partial_j\left(N\sqrt{h}h^{ij}\right)\partial_i\phi + \\
& - N\sqrt{h}\frac{\partial V(\phi)}{\partial\phi} + \partial_i\left(N^i p_\phi\right).
\end{aligned} \tag{20}$$

The complete dynamics of the coupled gravito-scalar system is represented by the above dynamical equations together with equation (16) and the first and

second class constraints (14) and (15), which tell us we can not choose the fields and their conjugate momenta arbitrarily.

3 Physical interpretation of the "kinematical action"

We have introduced in the previous section the lapse-shift decomposition of the deformation vector (2). It is worth noting that we can obtain such equation varying an action built to this aim. It is the so-called kinematical action and takes the following form:

$$S = \int_{\Sigma^3 \times \Re} dt d^3x \left(p_\mu \partial_t y^\mu - N p_\mu n^\mu - N^i p_\mu e_i^\mu \right). \tag{21}$$

If we now vary the action (21) with respect to the dynamical variables p_μ and y^μ, and we put these two variations equal to zero, we obtain respectively:

$$\partial_t y^\mu = N n^\mu + N^i \partial_i y^\mu, \qquad \partial_t p_\mu = -N p_\rho \partial_\mu n^\rho + \partial_i \left(N^i p_\mu \right). \tag{22}$$

The first one of such equations is the lapse-shift decomposition of the deformation vector, while the second one provides the dynamical evolution for p_μ, which is the conjugate momenta to the vector y^ρ.

The kinematical action is used in quantum field theory on curved spacetime, in order to reparameterize the field action [6], [18], but it will be clear in the next section how, in our approach, it plays an important role also in the reformulation of the canonical quantum gravity.

In this section we want to investigate the physical meaning of the "kinematical term", which will outline either the main aspects of our reformulation of the canonical quantum gravity, either the meaning of the reparameterization in quantum field on curved space.

To get the searched physical insight, let us rewrite the equations (22) in a covariant form. To this aim we recall to denote the coordinates (t, x^i) by barred Greek indices: $\overline{\mu}, \overline{\nu}, \overline{\rho}$.... and we also remark that the following relations take place: $\partial_t = \partial_t y^\mu \partial_\mu$, $\partial_i = \partial_i y^\mu \partial_\mu$, $n^{\overline{\mu}} \partial_{\overline{\mu}} = n^\mu \partial_\mu$.

Now remembering that the normal vector n^μ has components $n^{\overline{\mu}} \equiv \left(\frac{1}{N}, -\frac{N^i}{N} \right)$ in the system $\left(t, x^i \right)$, it is possible to rewrite the first one of equations (22) in the following form: $n^\mu = n^{\overline{\rho}} \partial_{\overline{\rho}} y^\mu$; this equation ensures that, after the variation n^μ is a real unit time-like vector, i.e.

$$g_{\mu\nu} n^\mu n^\nu = g_{\mu\nu} n^{\overline{\rho}} \partial_{\overline{\rho}} y^\mu n^{\overline{\sigma}} \partial_{\overline{\sigma}} y^\nu = g_{\overline{\rho}\overline{\sigma}} n^{\overline{\rho}} n^{\overline{\sigma}} = -1, \tag{23}$$

the last equality being true by construction of $g_{\overline{\mu\nu}}$ and $n^{\overline{\mu}}$. Moreover, since n^μ is in any system of coordinates normal to the hypersurfaces Σ^3, then we see how the use of the kinematical action allows to overcome the ambiguity in the existence of a real time-like normal vector field, we have spoken about in the introduction of this paper.

Now using the relations $\partial_t = \partial_t y^\mu \partial_\mu$, $\partial_i = \partial_i y^\mu \partial_\mu$, $n^{\overline{\mu}}\partial_{\overline{\mu}} = n^\mu \partial_\mu$ and the first one of equations (22), we may rewrite the second kinematical equation, concerning the momentum dynamics as follows:

$$n^\rho \left[\partial_\rho \left(N p_\mu\right) - \partial_\mu \left(N p_\rho\right)\right] = -\partial_\mu \left(N p_\nu n^\nu\right) + p_\mu \left(n^\rho \partial_\rho N + \partial_i N^i\right); \tag{24}$$

we note that p_μ is not a vector, but it is a vector density of weight 1/2; thus we can rewrite it as $p_\mu = -\sqrt{h}\varepsilon\pi_\mu$, where ε is a real 3-scalar and π_μ is a vector, such that it satisfies the relation $n^\mu \pi_\mu = -1$. Using this new expression for p_μ, equation (24) rewrites:

$$\varepsilon n^\rho \left(\partial_\rho \pi_\mu - \partial_\mu \pi_\rho\right) = -\pi_\mu \frac{1}{\sqrt{-\overline{g}}} \partial_\rho \left(\sqrt{-\overline{g}}\varepsilon n^\rho\right), \tag{25}$$

which covariantly reads

$$\varepsilon n^\rho \left(\nabla_\rho \pi_\mu - \nabla_\mu \pi_\rho\right) + \pi_\mu \nabla_\rho \left(\varepsilon n^\rho\right) = 0. \tag{26}$$

Then, multiplying equation (26) for n^μ, we get

$$\nabla_\rho \left(\varepsilon n^\rho\right) = 0. \tag{27}$$

A perfect fluid, having entropy density σ and 4-velocity u_μ, satisfies the equation $\nabla_\mu \left(\sigma u^\mu\right) = 0$, but for a dust case the density of entropy is proportional to the density of energy ($\sigma \propto \varepsilon$), so that equation (27) is the one for a dust fluid of density of energy ε and 4-velocity n_μ.

Now, using equation (27), we can rewrite the relation (26) as

$$n^\rho \left(\nabla_\rho \pi_\mu - \nabla_\mu \pi_\rho\right) = 0. \tag{28}$$

Setting now $\pi_\mu = n_\mu + s_\mu$, with $n^\mu s_\mu = 0$, from above, we arrive to

$$n^\rho \nabla_\rho n_\mu = n^\rho \left(\nabla_\mu s_\rho - \nabla_\rho s_\mu\right) = \gamma n^\rho F_{\mu\rho}, \tag{29}$$

with $s_\rho = \gamma A_\rho$, where γ is a constant and $F_{\mu\rho} = \nabla_\mu A_\rho - \nabla_\rho A_\mu$ (obviously $n^\rho A_\rho = 0$).

Thus equation (29), together with (27) are the field equations of a dust fluid with density of energy ε, whose 4-velocity n^μ is tangent to a space-time curve associated to the presence of an "electromagnetic-like" field (say a *gravito-electromagnetic field*). So, on a classical level, the kinematical action is equivalent to the action of such a dust fluid and, in this sense, it is upgrated from its geometrical nature to a physical state.

The condition $n^\rho A_\rho = 0$ can be written in the system $\left(t, x^i\right)$ as $n_{\overline{\rho}} A^{\overline{\rho}} = 0$, from which it follows $A^{\overline{0}} = 0$ and this means that in the fluid reference we have to do with a gauge condition such that $A^\mu \equiv (0, \underline{A})$, i.e. with a simple 3-vector potential for the gravito-electromagnetic field.

Now let us come back to the kinematical action (21): varying it with respect to N and N^i, we obtain the corresponding super-Hamiltonian and supermomentum of the kinematical term:

$$H^k = p_\mu n^\mu, \qquad H_i^k = p_\mu e_i^\mu, \tag{30}$$

Using the definitions above introduced for p_μ and s_μ we have:

$$H^k = \sqrt{h}\varepsilon, \qquad H_i^k = -\sqrt{h}\varepsilon\gamma A_\mu e_i^\mu. \tag{31}$$

It is clear that $A_\mu e_i^\mu = A_\mu \dfrac{\partial y^\mu}{\partial x^i}$ is a transformation of coordinates from the generic system y^μ to the system of the hypersurface, that is the one which we have before indicated with barred indices. So we write $A_\mu e_i^\mu = A_i$, that is we introduce the projection of the field A_μ on the spatial hypersurfaces.

So equations (31) rewrites as:

$$H^k = \sqrt{h}\varepsilon, \qquad H_i^k = -\sqrt{h}\varepsilon\gamma A_i. \tag{32}$$

In [24] is shown that the energy-momentum tensor of the dust is orthogonal to the hypersurfaces Σ^3; this is the reason why it contributes only to the super-Hamiltonian, by its energy density. Moreover, it is possible to show, via a simple model, why the presence of the field A_μ has, instead, effects only on the supermomentum. To this end, let us consider an interaction between a current j^μ and a field B_μ; then the Hamiltonian of interaction will be:

$$H_{int} = \int d^4x \sqrt{-g} j^\mu B_\mu. \tag{33}$$

Since H_{int} is obviously a scalar, we can rewrite it in the system of coordinates with barred indices, as follows

$$H_{int} = \int d^4\overline{x} N \sqrt{h} j^{\overline{\mu}} B_{\overline{\mu}}; \tag{34}$$

taking now $j^{\overline{\mu}} = \varepsilon n^{\overline{\mu}}$ (current of matter) and $B_{\overline{\mu}} = \gamma A_{\overline{\mu}}$, we have, remembering also that $n^{\overline{\mu}} \equiv \left(\dfrac{1}{N}, -\dfrac{N^i}{N}\right)$,

$$H_{int} = \int d^4\overline{x} \sqrt{h}\varepsilon\gamma \left(A_{\overline{0}} - N^{\overline{i}} A_{\overline{i}}\right). \tag{35}$$

This expression no more depends on the lapse function, so that it does not contribute to the super-Hamiltonian, while the contribution to the supermomentum is just the one in equation (31).

Above we have introduced the projection of the field A_μ on the spatial hypersurfaces, i.e. $A_i = A_\mu e^\mu_i$; this is of course a simple transformation of coordinates, but it does not assure A_i is a 3-vector. To show this, we define $A^i = A^\mu e^i_\mu$; it is worth noting that it is not a transformation of coordinates, but this choice on how to project the contravariant 4-vector A^μ, is sufficient to show that $A_i = h_{ik}A^k$, which ensures A_i is a 3-vector on the hypersurfaces, which lowers and raises its index by the induced 3-metric.
In fact starting from the expression of A_i and recalling that $e^\mu_i = h_{ik}g^{\mu\nu}e^k_\nu$,, we can write:

$$A_i = A_\mu e^\mu_i = A_\mu h_{ik}g^{\mu\nu}e^k_\nu = h_{ik}A^\nu e^i_\nu = h_{ik}A^k, \tag{36}$$

where, in the last equality, we have used the definition of A^k.

To conclude this section, we want to study the behaviors of ε and A_i; to this end we start from equations (22), multiplying the second one by n^μ, and remembering that $n^\mu\partial_\mu = n^{\overline{\mu}}\partial_{\overline{\mu}}$, we arrive to

$$\partial_t\left(\sqrt{h}\varepsilon\right) - \partial_i\left(\sqrt{h}\varepsilon N^i\right) = 0; \tag{37}$$

Moreover, by multiplying the second one with e^μ_i and considering also the first kinematical equation, we get an expression of the form:

$$\partial_t\left(\sqrt{h}\varepsilon\gamma A_i\right) - \partial_k\left(\sqrt{h}\varepsilon\gamma N^k A_i\right) = \sqrt{h}\varepsilon\gamma A_k\partial_i N^k - \sqrt{h}\varepsilon\partial_i N. \tag{38}$$

To treat these two equations (37) and (38) in a general reference frame, it is a very difficult task, but it becomes very simple in a synchronous reference, where $N = 1$ and $N^i = 0$; in this particular case we have:

$$\partial_t\left(\sqrt{h}\varepsilon\right) = 0, \qquad \partial_t\left(\sqrt{h}\varepsilon\gamma A_i\right) = 0. \tag{39}$$

The first one of the above equations means that $\sqrt{h}\varepsilon = -\omega\left(x^i\right)$ where ω is a scalar density of weight 1/2, which depends only on x^i; we note that $\varepsilon = -\dfrac{\omega\left(x^i\right)}{\sqrt{h}}$, this means ε is the density of energy of a non relativistic dust. While from the second one we obtain $\gamma A_i\omega\left(x^i\right) = -k_i\left(x^k\right)$, which is a 3-vector density of weight 1/2 and depends only on x^i (we have to do with a simple magnetic term). It is clear that we can now write the super-Hamiltonian and super-momentum of the kinematical term as follows

$$H^k = -\omega\left(x^l\right) \qquad H^k_i = -k_i\left(x^l\right) \tag{40}$$

We will return on the above expression in the next section, when treating the eigenvalues problem and the classical limit of the quantized theory; indeed we will find a connection between the density of energy of the dust and the eigenvalue of the super-Hamiltonian operator as well as between the eigenvalues of the super-momentum operator and the presence of the field A_i.

4 Canonical quantization of the model

Our reformulation of the canonical quantum gravity is based on a fundamental criticism about the possibility to speak of a unit time-like normal field and of space-like hypersurfaces, which are at the ground of the ADM formalism, when referring to a quantum space-time; in fact, in this case, either the time-like nature of a vector field, either the space-like nature of the hypersurfaces can be recognized at most in average sense, i.e. with respect to expectation values. This consideration makes extremely ambiguous to apply the 3+1 splitting on a quantum level and leads us to claim that the canonical quantization of gravity has sense only when referred to a fixed slicing, or in other words, when referred to a fixed reference frame, i.e. only after the notion of space and time are physically distinguishable. To fix the slicing we have to choose a particular family of hypersurfaces and this means we have to fix the lapse function N and the shift vector N^i. However, so doing, we loose the Hamiltonian constraints (14), (15) and, with them, a standard procedure to quantize the dynamics of the system; as a solution to this problem, we propose to introduce, like in the fixed background field theory, the kinematical action [18] to the total gravity-matter one, to reparameterize the action; hence we obtain:

$$S^{g\phi k} = \int\limits_{\Sigma^3\times\Re} dtd^3x \left\{p^{ij}\partial_t h_{ij} + \pi\partial_t N + \pi_k\partial_t N^k + p_\phi\partial_t\phi + p_\mu\partial_t y^\mu + \right.$$
$$\left. - \left(\lambda\pi + \lambda^i\pi_i + N\left(H^g + H^\phi + H^k\right) + N^i\left(H^g_i + H^\phi_i + H^k_i\right)\right)\right\}. \quad (41)$$

Now the lapse function N and the shift vector N^i are to be regarded as dynamical variables, obtaining in this way the new Hamiltonian constraints, i.e.

$$\pi = 0, \qquad \pi_k = 0, \quad (42)$$
$$H^g + H^\phi + H^k = 0, \qquad H^g_i + H^\phi_i + H^k_i = 0. \quad (43)$$

We have, also, introduced the new dynamical field $y^\mu = y^\mu\left(t, x^i\right)$ and its conjugate momentum $p_\mu = p_\mu\left(t, x^i\right)$, whose variation leads to the kinematical equation (22).

Though from a mathematical point of view, to fix the reference frame is, in view of the reparameterization which restores the canonical constraints, a well defined procedure, it requires a physical interpretation; indeed the open question is: which are the physical consequences of fixing the slicing?
The complete answer to this question will be clear at the end of this section, but we can say already now that fixing the reference frame we modify the system: the dynamical equations and the constraints, which we can obtain by the variation of the action (41), describe no more the dynamics of the initial system formed by the gravitational and a scalar field, but the coupled system of these two fields with a non relativistic dust fluid which interacts with the field A_i we have

studied in the previous section. We remark that in a purely classical system it is not necessary to introduce this additional term to the gravitaty-matter action and therefore we expect that the dust has not effect on the dynamics of such systems have no undergone a classical limit (but the non relativistic dust becomes important in the description of those systems which evolve from a quantum state).

Now to quantize the new constraints (42), (43) we use the canonical procedure, by implementing the canonical variables to quantum operators, i.e. $Q \to \widehat{Q}$, $P \to \widehat{P}$; here Q and P denote, respectively, a generic field and its conjugate momentum, which satisfy the usual relations of commutation, i.e.

$$\left[\widehat{Q}\left(x\right), \widehat{P}\left(y\right)\right] = i\hbar\delta\left(x-y\right), \tag{44}$$

in the functional representation we have

$$\widehat{Q}\left(x\right) \longrightarrow Q\left(x\right), \qquad \widehat{P}\left(y\right) \longrightarrow -i\hbar\frac{\delta}{\delta Q\left(x\right)}, \tag{45}$$

where the operators are applied to a wave functional $\psi = \psi\left(Q\right).$

We assume that the state of the gravitational and matter system be described by a wave functional $\Psi = \Psi\left(y^{\mu}, \phi, h_{ij}, N, N^{i}\right)$. Then the new quantum dynamics of the whole system is now described by the functional differential system:

$$\frac{\delta\Psi}{\delta N} = 0, \qquad\qquad \frac{\delta\Psi}{\delta N^{i}} = 0, \tag{46}$$

$$i\hbar n^{\mu}\frac{\delta\Psi}{\delta y^{\mu}} = \left(\widehat{H}^{g} + \widehat{H}^{\phi}\right)\Psi, \qquad i\hbar\partial_{i}y^{\mu}\frac{\delta\Psi}{\delta y^{\mu}} = \left(\widehat{H}^{g}_{i} + \widehat{H}^{\phi}_{i}\right)\Psi, \tag{47}$$

being $\widehat{H}^{g} + \widehat{H}^{\phi}$and $\widehat{H}^{g}_{i} + \widehat{H}^{\phi}_{i}$ the Hamiltonian operators after the quantum implementation of the canonical variables. The first line of the above equations tell us that the wave functional does not depend on the lapse function N and the shift vector N^{i}, so, since now, we limit our attention on the other two equations, considering that the wave functional Ψ depends only on the 3-metric $h_{ij}\left(x^{k}\right)$, the scalar field $\phi\left(x^{k}\right)$ and the new field $y^{\mu}\left(x^{k}\right)$, which plays the role of the time variable, by specifying the hypersurface on which the wave functional is taken (we stress how its spatial gradients behaves like potential terms).

Moreover, the second of equation (47) ensures the invariance of the wave functional under the spatial diffeomorphism and then, denoting by the notation $\{h_{ij}\}$ a whole class of 3-geometries (i.e. connected via 3-coordinates reparameterization), the wave functional should be taken on such more appropriate variable instead of a special realization of the 3-metric.

In the first of equations (47) the vector field $n^{\mu}\left(y^{\rho}\right)$ is an arbitrary one without any peculiar geometrical meaning; but when taking into account the first of kinematical equation (22), n^{μ} becomes a real unit normal vector field, since, once fixed N and N^{i}, $y^{\mu}\left(t, x^{i}\right)$ pays the price for its geometrical interpretation.

These considerations lead us to claim that the first of equation (22) should be included in the dynamics even on the quantum level. The physical justification for this statement relies on the fact that no information about the dynamic of the kinematical dust comes from such an equation has discussed in the previous section; in fact there we have shown how the whole "hydrodynamics" of the dust be entirely contained in the momentum equation. In agreement to what we said in the introduction to this work, the surviving of this classical equation on a quantum level, reflects the classical nature of the "device" operating the (3 + 1)-splitting.
To take into account this equation is equivalent to reduce y^μ to a simple $\infty-$dimensional parameter for the system dynamics.

In agreement with this point of view, we can smear the quantum dynamics on a whole 1-parameter family of spatial hypersurfaces Σ_t^3 filling the space-time; as soon as we introduce the notation

$$\partial_t = \int_{\Sigma_t^3} d^3x \partial_t y^\mu \frac{\delta}{\delta y^\mu}, \tag{48}$$

then equations (47) acquire the Schrödinger form

$$i\hbar\partial_t\Psi = \widehat{\mathcal{H}}\Psi, \tag{49}$$

where

$$\widehat{\mathcal{H}} = \int_{\Sigma_t^3} d^3x \left[N\left(\widehat{H}^g + \widehat{H}^\phi\right) + N^i\left(\widehat{H}_i^g + \widehat{H}_i^\phi\right)\right]. \tag{50}$$

In this new framework the wave functional can be taken directly on the label time (i.e. $\Psi = \Psi\left(t, \phi, h_{ij}\right)$) (where we have removed the curl bracket from h_{ij} because, when regarded in such a manner, the wave functional is no longer invariant under 3-diffeomorphism), since the latter becomes a physical clock via the correspondence, we show below, between the eigenvalue problem of the equation (49) and the energy-momentum of the dust discussed in the previous section.

In order to construct the Hilbert space associated to the Schrödinger-like equation we must prove the Hermitianity of the Hamiltonian operator; since the Hermitianity of the ϕ term was proven in [18], as well as the Hermitianity of the operator $\widehat{H}^g$ in [24] under the following choice for the normal ordering

$$G_{ijkl}p^{ij}p^{kl} \rightarrow -\hbar^2 \frac{\delta}{\delta h_{ij}}\left(G_{ijkl}\frac{\delta\left(...\right)}{\delta h_{kl}}\right), \tag{51}$$

then it remains to be shown the Hermitianity of the operator $\widehat{h} = \int_{\Sigma_t^3} d^3x N^i \widehat{H}_i^g$.

In Dirac notation we have to show that:

$$\left\langle \Psi_1 \left| \widehat{h} \right| \Psi_2 \right\rangle = \left\langle \Psi_2 \left| \widehat{h} \right| \Psi_1 \right\rangle^*. \tag{52}$$

To this aim we write down the explicit expression of the above bracket:

$$\left\langle \Psi_1 \left| \widehat{h} \right| \Psi_2 \right\rangle = 2i\hbar \int_{\mathcal{F}_t} Dh \int_{\Sigma_t^3} d^3x \Psi_1^* N^i h_{ik} \nabla_j \frac{\delta}{\delta h_{kj}} \Psi_2, \tag{53}$$

where Dh is the Lebesgue measure in the 3-geometries functional space.

Now integrating by parts, considering that the hypersurfaces Σ_t^3 are compact and using, in view of the functional Gauss theorem, the following relation:

$$\int_{\mathcal{F}_t} Dh \int_{\Sigma_t^3} d^3x \frac{\delta}{\delta h_{kj}} (......) = 0, \tag{54}$$

we can rewrite the expression (53) in the following form:

$$\left\langle \Psi_1 \left| \widehat{h} \right| \Psi_2 \right\rangle = 2i\hbar \int_{\mathcal{F}_t} Dh \int_{\Sigma_t^3} d^3x \frac{\delta}{\delta h_{kj}} \left(\Psi_1^* \left(\nabla_j N^i \right) h_{ik} \right) \Psi_2. \tag{55}$$

It is possible to show that two of the terms, which come from the right side of (55) when the functional derivative operates on the quantities in the parenthesis, are zero. In fact, acting with the functional derivative on the 3-metric, we obtain:

$$2i\hbar \int_{\mathcal{F}_t} Dh \int_{\Sigma_t^3} d^3x \Psi_1^* \left(\nabla_j N^i \right) \frac{\delta h_{ik}}{\delta h_{kj}} \Psi_2 = -2i\hbar \int_{\mathcal{F}_t} Dh \int_{\Sigma_t^3} d^3x \nabla_j \left(\Psi_1^* \Psi_2 \right) N^j, \tag{56}$$

where we have integrated by parts and used the compactness of the hypersurfaces Σ_t^3. But the right hand side of (56) is zero, because Ψ is a functional, so it does not depend on x.

When the functional derivative in expression (55) acts on the covariant derivative of the shift vector, we obtain:

$$\begin{aligned} &2i\hbar \int_{\mathcal{F}_t} Dh \int_{\Sigma_t^3} d^3x h_{ik} \Psi_1^* \Psi_2 \frac{\delta}{\delta h_{kj}} \left(\nabla_j N^i \right) = \\ &\qquad = 2i\hbar \int_{\mathcal{F}_t} Dh \int_{\Sigma_t^3} d^3x h_{ik} \Psi_1^* \Psi_2 \frac{\delta}{\delta h_{kj}} \left(\Gamma^i_{jm} N^m \right), \end{aligned} \tag{57}$$

since in the right side term, the derivative operator is applied to a function of x and not to a functional, thus, like in the case of the variation with respect a dynamical variable, the ordinary derivative operator and the functional one commute, so it is simple to show that $\frac{\delta}{\delta h_{kj}} \left(\Gamma^i_{jm} N^m \right) = 0$, thus the term (57) is identically zero.

Finally the expression (55) can be rewrite:

$$\left\langle \Psi_1 \left| \widehat{h} \right| \Psi_2 \right\rangle = 2i\hbar \int\limits_{\mathcal{F}_t} Dh \int\limits_{\Sigma_t^3} d^3x \frac{\delta \Psi_1^*}{\delta h_{kj}} \left(\nabla_j N^i \right) h_{ik} \Psi_2 =$$

$$= -2i\hbar \int\limits_{\mathcal{F}_t} Dh \int\limits_{\Sigma_t^3} d^3x \Psi_2 N^i h_{ik} \nabla_j \frac{\delta \Psi_1^*}{\delta h_{kj}} = \left\langle \Psi_2 \left| \widehat{h} \right| \Psi_1 \right\rangle^* . \tag{58}$$

The above equality assures $\widehat{\mathcal{H}}$ is an Hermitian operator.

Defining the following inner product:

$$\langle \Psi_1 \mid \Psi_2 \rangle = \int\limits_{y_t} DhD\phi \Psi_1^* \Psi_2, \tag{59}$$

where $DhD\phi$ is the Lebesgue measure for the functional space of all the dynamical variables and y_t is the corresponding functional domain, we can turn the space of solutions of the Schrödinger-like equation into an Hilbert space. We interpret the above bracket as the probability that a state $|\Psi_1\rangle$ falls into another state $|\Psi_2\rangle$ and, defining the density of probability $\rho = \Psi^* \Psi$, we can also construct the amplitude for the system lying in a field configuration. By the Hermitian character of the operator $\widehat{\mathcal{H}}$, it is possible to show that the probability is constant in time, in fact:

$$\partial_t \langle \Psi_1 | \, \Psi_2 \rangle = \int\limits_{\Sigma_t^3} d^3x \partial_t y^\mu \frac{\delta}{\delta y^\mu} \langle \Psi_1 | \, \Psi_2 \rangle = \frac{i}{\hbar} \left(\left\langle \widehat{\mathcal{H}} \Psi_1 \mid \Psi_2 \right\rangle - \left\langle \Psi_1 \mid \widehat{\mathcal{H}} \Psi_2 \right\rangle \right) = 0, \tag{60}$$

and the general character of the deformation vector allows us to write the fundamental conservation law

$$\frac{\delta \langle \Psi_1 | \, \Psi_2 \rangle}{\delta y^\mu} = 0, \tag{61}$$

which assures the probability does not depend on the choice of the hypersurface.

The density of probability ρ satisfies a continuity equation, which can be obtained multiplying the Schrödinger-like equation times the complex conjugate wave function Ψ^* and the complex conjugate equation times the wave function Ψ, i.e.

$$i\hbar \Psi^* \partial_t \Psi = \Psi^* \widehat{\mathcal{H}} \Psi, \qquad -i\hbar \Psi \partial_t \Psi^* = \Psi \widehat{\mathcal{H}}^* \Psi^*, \tag{62}$$

subtracting the second of equation (62) from the first one, we obtain:

$$
\begin{aligned}
i\hbar\partial_t\left(\Psi\Psi^*\right) = \int\limits_{\Sigma_t^3} d^3x \Bigg\{ & -\hbar^2\left(\Psi^* N\frac{\delta}{\delta h_{ij}}G_{ijkl}\frac{\delta}{\delta h_{kl}}\Psi - \Psi N\frac{\delta}{\delta h_{ij}}G_{ijkl}\frac{\delta}{\delta h_{kl}}\Psi^*\right) + \\
& - \hbar^2\left(\Psi^*\frac{N}{2\sqrt{h}}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\Psi - \Psi\frac{N}{2\sqrt{h}}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\Psi^*\right) + \\
& + 2i\hbar\left(\Psi^* N^i h_{ik}\nabla_j\frac{\delta}{\delta h_{kj}}\Psi + \Psi N^i h_{ik}\nabla_j\frac{\delta}{\delta h_{kj}}\Psi^*\right) + \\
& - i\hbar\left(\Psi^* N^i\partial_i\phi\frac{\delta}{\delta\phi}\Psi + \Psi N^i\partial_i\phi\frac{\delta}{\delta\phi}\Psi^*\right)\Bigg\}, \qquad (63)
\end{aligned}
$$

defining now the tensor probability current A_{ij}, which is connected with the 3-metric tensor field, in the following way:

$$
A_{ij} = -i\hbar\left(\Psi^* NG_{ijkl}\frac{\delta}{\delta h_{kl}}\Psi - \Psi NG_{ijkl}\frac{\delta}{\delta h_{kl}}\Psi^*\right) + 2h_{ki}\left(\nabla_j N^k\right)\Psi^*\Psi, \quad (64)
$$

and the scalar probability current A, connected, instead, to the presence of the scalar field ϕ, as:

$$
A = -i\hbar\frac{N}{2\sqrt{h}}\left(\Psi^*\frac{\delta}{\delta\phi}\Psi - \Psi\frac{\delta}{\delta\phi}\Psi^*\right) - i\hbar\left(\phi\partial_i N^i\Psi^*\Psi\right), \quad (65)
$$

the equation (63) takes the following form:

$$
\partial_t\rho + \int\limits_{\Sigma_t^3} d^3x\left(\frac{\delta A_{ij}}{\delta h_{ij}} + \frac{\delta A}{\delta\phi}\right) = 0, \quad (66)
$$

integrating on the functional space y_t, using the generalized Gauss theorem (54), the continuity equation assures that the probability is constant in time as above.

Let us now reconsider the Schrödinger dynamics in terms of a time independent eigenvalues problem. To this end we expand the wave functional as follows:

$$
\begin{aligned}
\Psi\left(t,\phi,h_{ij}\right) = \int\limits_{y_t^*} D\Omega DK\Theta\left(\Omega,K_i\right)\chi_{\Omega,K_i}\left(\phi,h_{ij}\right)\cdot \\
\cdot\exp\left\{-\frac{i}{\hbar}\int\limits_{t_0}^{t}dt'\int\limits_{\Sigma_t^3}d^3x\left(N\Omega + N^iK_i\right)\right\}, \qquad (67)
\end{aligned}
$$

being t_0 an assigned initial "instant". Where $D\Omega DK$ denotes the Lebesgue measure in the functional space y_t^* of the conjugate function $\Omega\left(x^i\right)$ and $K_i\left(x^i\right)$, $\Theta = \Theta\left(\Omega,K_i\right)$ a functional valued in this domain, whose form is determined by

the initial conditions $\Psi_0 = \Psi(t_0, \phi, h_{ij})$. When we substitute the expansion (67) of the wave functional into the equation (49), it is satisfied only if take place the following ∞^3-dimensional eigenvalues problem:

$$\left(\widehat{H}^g + \widehat{H}^\phi\right)\chi_{\Omega,K_i} = \Omega\left(x^j\right)\chi_{\Omega,K_i}, \qquad \left(\widehat{H}^g_i + \widehat{H}^\phi_i\right)\chi_{\Omega,K_i} = K_i\left(x^j\right)\chi_{\Omega,K_i}. \tag{68}$$

Now to characterize the physical meaning of the above eigenvalues, we construct the semi-classical limit of the Schrödinger-like equation, by splitting the wave functional into its modulus and phase, as follows:

$$\Psi = \sqrt{\rho}e^{\frac{i}{\hbar}\sigma} \tag{69}$$

Then in the limit $\hbar \to 0$ we obtain for σ an Hamilton-Jacobi equation of the form:

$$\begin{aligned} -\partial_t\sigma = &\int_{\Sigma^3_t} d^3xN\left(G_{ijkl}\frac{\delta\sigma}{\delta h_{ij}}\frac{\delta\sigma}{\delta h_{ij}} - \sqrt{h}^{(3)}R + \right.\\ &\left. + \frac{1}{2\sqrt{h}}\frac{\delta\sigma}{\delta\phi}\frac{\delta\sigma}{\delta\phi} + \frac{\sqrt{h}}{2}h^{ij}\partial_i\phi\partial_j\phi + \sqrt{h}V(\phi)\right) + \\ &- \int_{\Sigma^3_t} d^3xN^i\left(2h_{ik}\nabla_j\frac{\delta\sigma}{\delta h_{kj}} - \partial_i\phi\frac{\delta\sigma}{\delta\phi}\right) \end{aligned} \tag{70}$$

The non vanishing of the σ time derivative reflects the evolutive character appearing in the constructed theory and makes account for the presence, on the classical limit, of the dust matter discussed in the previous section. To clarify this feature, we set

$$\sigma(t, \phi, h_{ij}) = \tau(\phi, h_{ij}) + \int_{t_0}^{t} dt'\int_{\Sigma^3_t} d^3x\left(N\Omega + N^iK_i\right). \tag{71}$$

When we substitute this expression in the Hamilton-Jacobi equation, and identify $p^{ij} = \dfrac{\delta\tau}{\delta h_{ij}}$, $p_\phi = \dfrac{\delta\tau}{\delta\phi}$, then the equation (70) becomes equivalent to the $\infty-$dimensional ones:

$$\left(H^g + H^\phi\right) = \Omega\left(x^j\right) \qquad \left(H^g_i + H^\phi_i\right) = K_i\left(x^j\right) \tag{72}$$

We stress how these equations coincides with those ones obtainable by the eigenvalues problem (68), as soon as we choose the classical limit of $\chi \sim e^{\frac{i}{\hbar}\tau}$ thus, at the end of this analysis, recalling expressions (40) and (43), we can identify the super-Hamiltonian eigenvalue Ω with ω and the super-momentum eigenvalues K_i with k_i. On the other hand by equations (40) and (32), the above identification implies that: $\Omega = -\sqrt{h}\varepsilon$ and $K_i = -\gamma A_i\omega$.

The relation we obtained show how super-Hamiltonian and super-momentum eigenvalues are directly connected with the dust fields introduced in section 3. Even starting from a quantum point of view we recognize the existence of a dust fluid playing the role of a physical clock for the gravity-matter dynamics.

5 The Ehrenfest theorem

In this section we face the problem to derive an Ehrenfest approach for the expectation values dynamics, to this end we start by the natural relation:

$$\frac{d}{dt}\left\langle \widehat{T} \right\rangle = \frac{1}{i\hbar}\left\langle \left[\widehat{T}, \widehat{\mathcal{H}}\right] \right\rangle \tag{73}$$

where $\widehat{T}$ denotes a generic operator, acting on physical states. Now we implement this formula for the case of the 3-metric $\widehat{h}_{ij}$ and its conjugate momentum $\widehat{p}^{ij}$ as well as for the scalar field ϕ and its conjugate momentum p_ϕ; in view of the canonical equal time commutation relations:

$$[\phi, F(\phi)] = [p_\phi, G(p_\phi)] = 0 \tag{74}$$

$$[\phi(x), p_\phi(y)] = i\hbar\delta(x-y) \tag{75}$$

$$\left[\widehat{h}_{ij}(x), F\left(\widehat{h}_{kl}(y)\right)\right] = \left[\widehat{p}^{ij}(x), G\left(\widehat{p}^{kl}(y)\right)\right] = 0 \tag{76}$$

$$\left[\widehat{h}_{ij}(x), \widehat{p}^{kl}(y)\right] = i\hbar\delta_{ij}^{kl}\delta(x-y) \tag{77}$$

where F and G are two generic function; we obtain the following equations for the expectation value of the 3-metric field:

$$\begin{aligned}\frac{d}{dt}\left\langle \widehat{h}_{ab}(x) \right\rangle = \frac{1}{i\hbar}\int d^3y \Big\{ & N\left\langle \left[\widehat{h}_{ab}(x), \widehat{p}^{ij}(y)\right] \widehat{G}_{ijkl}(y)\,\widehat{p}^{kl}(y) \right\rangle + \\ & + N\left\langle \widehat{p}^{ij}(y)\,\widehat{G}(y)_{ijkl}\left[\widehat{h}_{ab}(x), \widehat{p}^{kl}(y)\right] \right\rangle + \\ & + 2\nabla_j N_i \left\langle \left[\widehat{h}_{ab}(x), \widehat{p}^{ij}(y)\right] \right\rangle \Big\}.\end{aligned} \tag{78}$$

Now expliciting the commutators, the above expression rewrites as:

$$\begin{aligned}\frac{d}{dt}\left\langle \widehat{h}_{ab}(x) \right\rangle = & N\left\langle \left(\widehat{G}_{abkl}(x)\,\widehat{p}^{kl}(x) + \widehat{p}^{ij}(x)\,\widehat{G}(x)_{ijab}\right) \right\rangle + \\ & + 2\left\langle \nabla_{(a}N_{b)} \right\rangle.\end{aligned} \tag{79}$$

Using the commutation relation, after few algebra, we obtain:

$$\frac{d}{dt}\left\langle \widehat{h}_{ab}\left(x\right)\right\rangle = 2N\left\langle \widehat{G}_{abkl}\left(x\right)\widehat{p}^{kl}\left(x\right)\right\rangle + \\ + 2\left\langle \nabla_{(a}N_{b)}\right\rangle - i\hbar\frac{3}{4}N\left\langle \left(\widehat{h}\left(x\right)\right)^{-1/2}\widehat{h}_{ab}\left(x\right)\right\rangle \tag{80}$$

For what regards the conjugate momentum operator, we have:

$$\frac{d}{dt}\left\langle \widehat{p}^{ab}\left(x\right)\right\rangle = \frac{1}{i\hbar}\int d^3y\left\{N\left\langle \widehat{p}^{ij}\left(y\right)\left[\widehat{p}^{ab}\left(x\right),\widehat{G}_{ijkl}\left(y\right)\right]\widehat{p}^{kl}\left(y\right)\right\rangle + \right. \\ - N\left\langle\left[\widehat{p}^{ab}\left(x\right),\widehat{h}^{1/2}\left(y\right)^{(3)}\widehat{R}\left(y\right)\right]\right\rangle + 2\left\langle \widehat{p}^{ij}\left(y\right)\left[\widehat{p}^{ab}\left(x\right),\nabla_j\right]N_i\right\rangle + \\ + \frac{N}{2}\left\langle \widehat{p}_\phi^2\left(y\right)\left[\widehat{p}^{ab}\left(x\right),\widehat{h}^{-1/2}\left(y\right)\right]\right\rangle + \\ + N\left\langle\left(\frac{1}{2}h^{ij}\left(y\right)\partial_i\widehat{\phi}\left(y\right)\partial_j\widehat{\phi}\left(y\right) + \widehat{V}\left(\phi\right)\right)\left[\widehat{p}^{ab}\left(x\right),\widehat{h}^{1/2}\left(y\right)\right]\right\rangle \tag{81}$$

expliciting the commutators, with a certain algebra, we obtain:

$$\frac{d}{dt}\left\langle p^{ab}\right\rangle = \frac{1}{2}\left\langle N\widehat{h}^{-1/2}\widehat{h}^{ab}\left(\widehat{p}^{ij}\widehat{p}_{ij} - \frac{1}{2}\widehat{p}^2\right)\right\rangle - \left\langle 2N\widehat{h}^{-1/2}\left(\widehat{p}^{ai}\widehat{p}_i^b - \frac{1}{2}\widehat{p}\widehat{p}^{ab}\right)\right\rangle + \\ - \left\langle N\widehat{h}^{1/2}\left({}^{(3)}\widehat{R}^{ab} - \frac{1}{2}{}^{(3)}\widehat{R}\widehat{h}^{ab}\right)\right\rangle + \left\langle \widehat{h}^{1/2}\left(\nabla^a\nabla^b N - \widehat{h}^{ab}\nabla^i\nabla_i N\right)\right\rangle + \\ - \left\langle 2\nabla_i\left(\widehat{p}^{i(a}N^{b)}\right) + \nabla_i\left(N^i\widehat{p}^{ab}\right)\right\rangle + \\ + \left\langle \frac{1}{4}\widehat{h}^{-1/2}\widehat{h}^{ab}\widehat{p}_\phi^2 - \frac{1}{2}\widehat{h}^{1/2}\widehat{h}^{ab}\left(\frac{1}{2}h^{ij}\partial_i\widehat{\phi}\partial_j\widehat{\phi} + \widehat{V}\left(\phi\right)\right)\right\rangle + \\ - i\hbar\frac{3}{4}\left\langle N\widehat{h}^{-1/2}\left(\widehat{p}^{ab} - \frac{1}{2}\widehat{h}^{ab}\widehat{p}\right)\right\rangle. \tag{82}$$

Now to obtain the dynamical equations for the mean values of the scalar field ϕ and its conjugate momentum p_ϕ, we write like above:

$$\frac{d}{dt}\left\langle \widehat{\phi}\left(x\right)\right\rangle = \frac{1}{i\hbar}\int d^3y\frac{1}{2}\left\{\left\langle N\widehat{h}^{-1/2}\left(y\right)\left[\widehat{\phi}\left(x\right),\widehat{p}_\phi^2\left(y\right)\right]\right\rangle + \right. \\ \left. + \left\langle N^i\left[\widehat{\phi}\left(x\right),\widehat{p}_\phi\left(y\right)\right]\partial_i\widehat{\phi}\left(y\right)\right\rangle\right\}, \tag{83}$$

using the commutation relations we simply obtain:

$$\frac{d}{dt}\left\langle \widehat{\phi}\left(x\right)\right\rangle = \left\langle N\widehat{h}^{-1/2}\left(x\right)\widehat{p}_\phi\left(x\right)\right\rangle + \left\langle N^i\partial_i\widehat{\phi}\left(x\right)\right\rangle. \tag{84}$$

For what regards the conjugate momentum of the scalar field we have

$$\frac{d}{dt}\left\langle \widehat{p}_{\phi}\left(x\right)\right\rangle =\frac{1}{i\hbar}\int d^{3}y\left\{\left\langle \partial_{i}\left(-\frac{1}{2}N\widehat{h}^{1/2}\widehat{h}^{ij}\left(y\right)\partial_{j}\widehat{\phi}\left(y\right)\right)\left[\widehat{p}_{\phi}\left(x\right),\widehat{\phi}\left(y\right)\right]\right\rangle +\right.$$
$$\left.+\left\langle N\widehat{h}^{1/2}\left(y\right)\left[\widehat{p}_{\phi}\left(x\right),\widehat{V}\left(\phi\right)\right]\right\rangle -\left\langle \partial_{i}\left(N^{i}\widehat{p}_{\phi}\left(y\right)\right)\left[\widehat{p}_{\phi}\left(x\right),\widehat{\phi}\left(y\right)\right]\right\rangle \right\}, \tag{85}$$

by which the dynamical equation below follows

$$\frac{d}{dt}\left\langle p_{\phi}\left(x\right)\right\rangle =\left\langle N\widehat{h}^{1/2}\left(x\right)\widehat{h}^{ij}\left(x\right)\partial_{i}\partial_{j}\widehat{\phi}\left(x\right)\right\rangle +\left\langle \partial_{j}\left(N\widehat{h}^{1/2}\left(x\right)\widehat{h}^{ij}\left(x\right)\right)\partial_{i}\widehat{\phi}\left(x\right)\right\rangle +$$
$$-\left\langle N\widehat{h}^{1/2}\left(x\right)\frac{\partial V\left(\phi\right)}{\partial\phi}\right\rangle +\left\langle \partial_{i}\left(N^{i}\widehat{p}_{\phi}\left(x\right)\right)\right\rangle . \tag{86}$$

We observe that the equations we obtain for the expectation values of the 3-metric field and of its conjugate momentum differ from those classical ones (17) and (18), by a term which is due to the particular normal ordering, we must choose in the theory. However in the semiclassical limit $\hbar \rightarrow 0$, when the wave function Ψ is taken in the form (69), such terms vanish because do not contain functional derivatives acting on the wave functional. So in this limit we get exactly the classical equations, if the function ρ approaches to δ function and if the canonical variables are obtained from the Hamilton-Jacobi principal function. By other words the additional terms in $\hbar$ vanish in the semiclassical limit simply because, in such condition, to speak of normal ordering makes no longer sense.

It is worth noting that the Ehrenfest procedure here discussed does not allow to reproduce the expectation values for the Hamiltonian constraints, but this is a natural consequence of fixing the lapse function N and the shift vector N^i in the gravity-matter action. The impossibility to have the vanishing Hamiltonian expectation values in this way leads just to the quantum phenomenology for the dust energy-momentum.

6 Concluding remarks

We have presented a reformulation of the canonical quantization of geometrodynamics with respect to a fixed reference frame; the main result obtained, including the kinematical action in the global dynamics, is the characterization of an appropriate internal physical clock. In our theory the role of clock is played by the reference fluid, comoving with the 3-hypersurfaces and its presence is necessary to distinguish between space-like and time-like geometrical objects before the canonical quantization procedure.

The fluid shows its presence through a comoving (non-positive defined) density of energy and momentum, which we have characterized either from a classical either from a quantum point of view: classically it comes from having

introduced the kinematical action, but its real nature must be investigated in the classical limit of the eigenvalues equations.

Some aspects in this theory need to be further developed on to be completely understood. We think of fundamental importance be the study of the phenomenological implications which this theory can have in a cosmological level; indeed the density of the dust and the field A_i could have direct something to do with the *dark matter* component observed in the Universe.

To be applicable to a generic inhomogeneous gravitational system, the theory here presented has to be reduced, necessarily, to a formulation on a lattice; recently some interesting proposal has appeared to discretize a quantum constraint [9], [10]and they are of course relevant for the discretization of the present theory. A more direct approach can be obtained applying the Regge calculus [26], [27], to the 3-geometries on the spatial hypersurfaces.

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